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Short Communication

A convenient technique for evaluating angular frequency in some nonlinear oscillations

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Abstract

In this paper, a convenient technique for evaluating angular frequency in some nonlinear oscillations is proposed. It is well known that once the restoring force function is given beforehand, the period of motion can be determined by an integral. The angular frequency has a relation with the period of motion as well as the integral. One makes a little modification for the integrand in the integral and a change of variable. It is found that if the three divisions are chosen on the integration interval and the trapezoid quadrature rule is used, a higher accurate result for the angular frequency can be achieved. For the restoring force being an odd function, three numerical examples are presented. The eardrum-type oscillation is studied as well. Higher accurate results for the angular frequency are obtained in all those examples.

1. Introduction

Problems of nonlinear vibration in conservative systems have a long history [1–3]. The well known nonlinear oscillation of the Duffing equation is an example in this field. The governing equation for the problem was formulated in Refs. [1,2]. In the case of ε being a small parameter in the equation, the equation is solved by using the Lindstedt–Poincare technique, the method of multiple scales, and the method of averaging [1,2]. Almost all perturbation methods are based on small parameters so that the approximate solutions can be expressed in a series of small parameters. The limitation of the perturbation method is easily seen. Clearly, in the case of ε being a larger value, the perturbation method is no longer valid.

On the other hand, many nonlinear vibration problems were solved by using the harmonic balance method and other methods [4–10]. The merit of the harmonic balance method is to balance the coefficients of Fourier series in the governing equation of the nonlinear oscillation, once the assumed motion is substituted in the equation.

Since the advanced computer was not available in an earlier time, investigators had to pay attention to the hand-handled type solution. The so-called hand-handled type solution is defined such that the solution can be obtained by hand and very elementary computation using the calculator (not computer). We will point out some difficult points by the use of this type of solution. The following is an example.

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In an earlier time, the following equations for Duffing-harmonic oscillation were formulated and studied [9,10,14]

$$\frac{d^2u}{dt^2} + \frac{u^3}{1+u^2} = 0,$$

with the initial conditions

$$u|_{t=0} = A, \quad \left.\frac{du}{dt}\right|_{t=0} = 0$$

where u is the displacement, A is a constant given beforehand.

In the harmonic balance method, one generally assume the motion, saying, in the following forms [9,10]

$$u = A\cos(\omega t) \quad \text{or} \quad u = \frac{c_1\cos(\omega t)}{1 + c_2\cos(2\omega t)},$$

where ω is the angular frequency, and c_1, c_2 are the undetermined coefficients. After some manipulation, particularly, the usage of the harmonic balance method, three approximate solutions for the angular frequency ω were suggested [9,10].

It is found that the suggested solutions for the angular frequency ω have up to 1.5% percentage error in the case that the "A" value changes from 0.01 to 100, even though they have a good behavior at the range $A > 10$ or $A < 0.01$. This result can be easily realized by the following reason. The real motion must be complicated, and sometimes the motion is far different from the harmonic motion. In this case, one cannot get an accurate result from modeling the motion by some simple function, for example; only two degrees of freedom are involved in the vibration mode. One way for improving the accuracy for the problem is to choose more terms in the modeled function. However, if one chooses more terms in the vibration mode, one must meet more complicated nonlinear algebraic equation, which cannot be solved by a simple derivation.

For solving some nonlinear vibration problems, some techniques, for example, the target function technique and the multiple-parameters technique were suggested [11,12]. These techniques provide higher accurate solution. However, these techniques must depend on the iteration, and the target function technique must rely on the numerical integration of the ordinary differential equation (ODE). These techniques belong to the computer-computed type solution.

The aim of this paper is to obtain higher accurate results by using the hand-handled type solution. The investigated item is the angular frequency only. It is well known that once the restoring force function is given beforehand, the period of motion T_p can be defined by an integral (see Eqs. (5), (8) and (9) below). It is found that if the three divisions are chosen on the integration interval $(0, \pi/2)$ and the trapezoid quadrature rule is used, a higher accurate result the angular frequency can be achieved.

For the restoring force being an odd function, three numerical examples are presented. The eardrum-type oscillation is studied as well. Higher accurate results for the angular frequency are obtained in all those examples.

2. General analyses and the convenient technique for evaluating the angular frequency

In following analysis, the nonlinear oscillation is generally defined by [1,2,10]

$$\frac{d^2u}{dt^2} + f(u) = 0, \tag{1}$$

where u is the displacement. The imposed boundary conditions take the form

$$u|_{t=0} = A, \quad \left.\frac{du}{dt}\right|_{t=0} = 0, \tag{2}$$

where A is a positive value. If $f(u) = u$, we have a solution $u = A\cos(\omega t)$ with $\omega = 1$, which is the case of the harmonic motion.

After multiplying both sides of Eq. (1) by $2\,\mathrm{d}u$ and making integration, from condition (2) one will yield

$$v^2 + g(u) = g(A), \tag{3}$$

where

$$v = \frac{\mathrm{d}u}{\mathrm{d}t}, \tag{4}$$

$$g(u) = 2\int_0^u f(u)\,\mathrm{d}u. \tag{5}$$

Eq. (3) represents the energy conservation relation for the motion.

In the following analysis, the following Duffing oscillation is taken as an example

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u^3 = 0 \quad \left(\text{or with } f(u) = u^3 \text{ in Eq. (1)}\right). \tag{6}$$

In this case, from Eq. (5) we have

$$g(u) = \frac{u^4}{2}. \tag{7}$$

If the angular frequency of motion is denoted by ω, from Eqs. (3) and (4) and the trajectory of motion (Fig. 1), we will find $\mathrm{d}t/\mathrm{d}u = -1/\sqrt{g(A) - g(u)}$ (for $0 \leqslant \omega t \leqslant \pi$) and $\mathrm{d}t/\mathrm{d}u = 1/\sqrt{g(A) - g(u)}$ (for $\pi \leqslant \omega t \leqslant 2\pi$). From this relation, we can obtain the period of the motion T_p and the angular frequency ω from the following equation [1,2,10]:

$$T_p = \frac{2\pi}{\omega} = 2\int_{-A}^{A} \frac{\mathrm{d}u}{\sqrt{g(A) - g(u)}}. \tag{8}$$

The integral (8) can be integrated numerically by an available quadrature rule (see Appendix A and Ref. [13]).

Alternatively, we can define the following relation:

$$I_o = \frac{1}{\omega} = \frac{T_p}{2\pi}. \tag{9}$$

Therefore, we have

$$I_o = \frac{1}{\omega} = \frac{1}{\pi}\int_{-A}^{A} \frac{\mathrm{d}u}{\sqrt{g(A) - g(u)}}. \tag{10}$$

The integral defined by Eq. (10) can be integrated with sufficient accuracy referred to Appendix A.

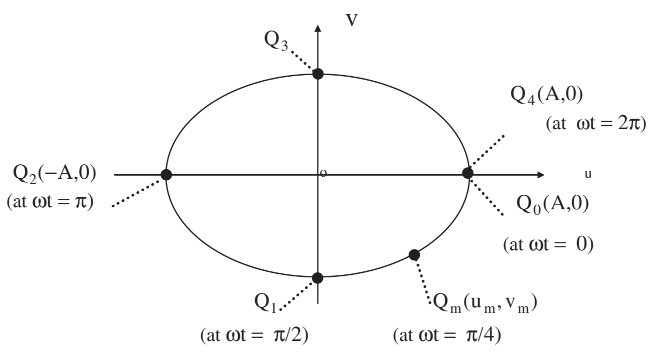

Fig. 1. The v vs. u trajectory for solution of the Duffing oscillation on the phase plane.

If $g(u)$ is an even function with respect to the argument u, the integral can be rewritten as

$$I_o = \frac{1}{\omega} = \frac{2}{\pi}\int_0^A \frac{\mathrm{d}u}{\sqrt{g(A)-g(u)}}. \tag{10a}$$

Further, The integral can be written alternatively

$$I_o = \frac{1}{\omega} = \frac{2}{\pi}\int_0^A \frac{h(u)\mathrm{d}u}{\sqrt{A^2-u^2}}, \tag{10b}$$

where

$$h(u) = \frac{\sqrt{A^2-u^2}}{\sqrt{g(A)-g(u)}}. \tag{11}$$

After letting $u = A\sin\theta$, we can define

$$h_1(\theta) = h(u)\big|_{u=A\sin\theta}. \tag{12}$$

After this substitution, we have

$$I_o = \frac{1}{\omega} = \frac{2}{\pi}\int_0^{\pi/2} h_1(\theta)\,\mathrm{d}\theta. \tag{10c}$$

In the case of $g(u) = u^4/2$, from Eqs. (7), (11) and (12), the explicit expression for the functions $h(u)$ and $h_1(\theta)$ are obtained

$$h(u) = \frac{\sqrt{2}}{\sqrt{A^2+u^2}}, \quad h_1(\theta) = \frac{\sqrt{2}}{A\sqrt{1+\sin^2\theta}}. \tag{13}$$

Two approximation schemes are suggested below. In the first scheme, we choose two divisions $(0, \pi/4)$ and $(\pi/4, \pi/2)$ for the interval $(0, \pi/2)$. After using the trapezoid quadrature rule (see Appendix B or Ref. [13]), from Eq. (10c) we will find

$$I_o = \frac{1}{\omega} = \frac{1}{4}\left[h_1(0) + 2h_1(\pi/4) + h_1(\pi/2)\right]. \tag{14}$$

Substituting $h_1(0) = \sqrt{2}/A$, $h_1(\pi/4) = 2/(\sqrt{3}A)$and $h_1(\pi/2) = 1/A$ in the above-mentioned equation yields

$$I_o = \frac{1}{\omega} = \frac{1}{4A}\frac{\sqrt{6}+4+\sqrt{3}}{\sqrt{3}}. \tag{14a}$$

Similarly, in the second scheme, we choose three divisions $(0, \pi/6)(\pi/6, \pi/3)$, and $(\pi/3, \pi/2)$ on the interval $(0, \pi/2)$. After using the trapezoid quadrature rule (see Appendix B or Ref. [13]), from Eq. (10c) we will find

$$I_o = \frac{1}{\omega} = \frac{1}{6}\left[h_1(0) + 2h_1(\pi/6) + 2h_1(\pi/3) + h_1(\pi/2)\right]. \tag{15}$$

Substituting $h_1(0) = \sqrt{2}/A$, $h_1(\pi/6) = 2\sqrt{2}/(\sqrt{5}A)$, $h_1(\pi/3) = 2\sqrt{2}/(\sqrt{7}A)$ and $h_1(\pi/2) = 1/A$ in the above-mentioned equation yields

$$I_o = \frac{1}{\omega} = \frac{1}{6A}\left(\sqrt{2} + \frac{4\sqrt{2}}{\sqrt{5}} + \frac{4\sqrt{2}}{\sqrt{7}} + 1\right). \tag{15a}$$

Note that, Eqs. (10b) and (10c) can be used to any case, only if the function $g(u)$ is an even function with respect to u. From Eq. (11) we see that when $u \to A$, $h(u)$ becomes 0/0 type expression. The value of $h(A)$ can be easily evaluated by L'hopital rule in calculus.

In the case of $f(u) = u^3$, the exact solution is [9,10,14]

$$\omega_{\mathrm{ex}} = \frac{\pi}{2F(1/\sqrt{2}, \pi/2)}A = 0.847202A, \tag{16}$$

where $F(1/\sqrt{2}, \pi/2)$ is the complete elliptic integral of the first kind.

If two-divisions technique is used, from Eq. (14a) we have the following angular frequency and percentage error:

$$\omega_{d2} = \frac{4\sqrt{3}}{\sqrt{6}+\sqrt{3}+4}A = 0.846809A, \quad \delta_{d2} = (\omega_{d2}-\omega_{ex})/\omega_{ex} = -0.0463\%, \tag{17}$$

where the subscript $d2$ means the result is obtained from two-divisions technique.

If three-divisions technique is used, from Eq. (15a) we have the following angular frequency and percentage error:

$$\omega_{d3} = \frac{6\sqrt{35}}{4(\sqrt{10}+\sqrt{14})+\sqrt{35}+\sqrt{70}}A = 0.847203A, \quad \delta_{d3} = (\omega_{d3}-\omega_{ex})/\omega_{ex} = 0.0002\%, \tag{18}$$

where the subscript $d3$ means the result is obtained from three-divisions technique.

The following two examples can be solved in a similar manner.

Example 2.1. In the first example, we consider the following Duffing oscillation:

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0 \quad (\text{or with } f(u) = u + \varepsilon u^3 \text{ in Eq. (1)}), \tag{19}$$

where ε is a parameter subject to variation. In this case, from Eqs. (5), (11) and (12), we have

$$g(u) = 2\int_0^u f(u)\,du = u^2 + \frac{\varepsilon u^4}{2},$$
$$h(u) = \frac{\sqrt{A^2-u^2}}{\sqrt{g(A)-g(u)}} = \frac{\sqrt{2}}{\sqrt{2+\varepsilon(A^2+u^2)}},$$
$$h_1(\theta) = \frac{\sqrt{2}}{\sqrt{2+\varepsilon A^2(1+\sin^2\theta)}}. \tag{20}$$

As before, the exact result for the angular frequency ω can be obtained from

$$T_p = \frac{2\pi}{\omega} = 2\int_{-A}^{A}\frac{du}{\sqrt{g(A)-g(u)}}, \tag{8}$$

$$I_o = \frac{1}{\omega} = \frac{2}{\pi}\int_0^{\pi/2} h_1(\theta)\,d\theta. \tag{10c}$$

In the example, we assume $A = 1$, and ε changes from 1,2, …, 10. The obtained exact results are expressed as

$$\omega_{ex} = F_1(\varepsilon). \tag{21}$$

This result is obtained from Eq. (8) and a quadrture rule described in Appendix A [13].

Similarly, if two-divisions technique shown by Eq. (14) is used, we have the following angular frequency:

$$\omega_{d2} = \frac{\sqrt{2}p_1p_2p_3}{p_2p_3 + 2p_3p_1 + p_1p_2}, \tag{22}$$

where

$$p_1 = \sqrt{8+4\varepsilon A^2}, \quad p_2 = \sqrt{8+6\varepsilon A^2}, \quad p_3 = \sqrt{8+8\varepsilon A^2}. \tag{23}$$

If three-divisions technique shown by Eq. (15) is used, we have the following angular frequency:

$$\omega_{d3} = \frac{3}{\sqrt{2}}\frac{q_1q_2q_3q_4}{(q_1+q_4)q_2q_3 + 2(q_2+q_3)q_1q_4}, \tag{24}$$

where

$$q_1 = p_1, \quad q_2 = \sqrt{8 + 5\varepsilon A^2}, \quad q_3 = \sqrt{8 + 7\varepsilon A^2}, \quad q_4 = p_3. \tag{25}$$

The computed results for ω_{ex}, ω_{d2} and ω_{d3} are listed in Table 1. From listed results we see that the maximum error for ω_{d2} (ω_{d3}) is −0.0281% (−0.0005%), respectively

Example 2.2. In the second example, we consider the following Duffing-harmonic oscillation [9,10,14]:

$$\frac{d^2u}{dt^2} + \frac{u^3}{1+u^2} = 0 \quad \left(\text{or with } f(u) = \frac{u^3}{1+u^2} \text{ in Eq. (1)}\right). \tag{26}$$

In this case, from Eqs. (5), (11) and (12), we have

$$g(u) = 2\int_0^u f(u)\,du = u^2 - \ln(1+u^2), \quad h(u) = \frac{\sqrt{A^2-u^2}}{\sqrt{A^2-u^2-\ln(1+A^2)+\ln(1+u^2)}},$$
$$h_1(\theta) = \frac{A\cos\theta}{\sqrt{A^2\cos^2\theta - \ln(1+A^2) + \ln(1+A^2\sin^2\theta)}}. \tag{27}$$

As before, the exact result for the angular frequency ω can be obtained from

$$T_p = \frac{2\pi}{\omega} = 2\int_{-A}^{A} \frac{du}{\sqrt{g(A)-g(u)}}, \quad (8)$$

$$I_o = \frac{1}{\omega} = \frac{2}{\pi}\int_0^{\pi/2} h_1(\theta)\,d\theta. \quad (10c)$$

In the example, we assume $A = 0.01$, 0.05,0.1, 10,50,100. The obtained exact results are expressed as

$$\omega_{ex} = F_2(A). \tag{28}$$

This result is obtained from Eq. (8) and a quadrture rule described in Appendix A [13].

Similarly, if two-divisions technique shown by Eq. (14) is used, we have the following angular frequency:

$$\omega_{d2} = \frac{4Ap_1p_2}{A^2(2p_1+p_2)+p_1p_2p_3}, \tag{29}$$

where

$$p_1 = \sqrt{A^2 - \ln(1+A^2)}, \quad p_2 = \sqrt{A^2 - 2\ln[(1+A^2)/(1+A^2/2)]}, \quad p_3 = \sqrt{A^2+1}. \tag{30}$$

Table 1
Comparison results for the angular frequencies ω_{ex}, ω_{d2} and ω_{d3} for the Duffing oscillation $d^2u/dt^2 + u(1+\varepsilon u^2) = 0$ with the conditions $u(0) = A$ and $u'(0) = 0$, for $\varepsilon = 1,2,\ldots,10$ and $A = 1$

ε	1	2	3	4	5	6	7	8	9	10
ω_{ex}	1.31778	1.56911	1.78442	1.97602	2.15042	2.31159	2.46217	2.60401	2.73849	2.86664
ω_{d2}	1.31776	1.56902	1.78424	1.97573	2.15004	2.31111	2.46160	2.60336	2.73776	2.86583
δ_{d2}	−0.00145	−0.00571	−0.01026	−0.01428	−0.01767	−0.02049	−0.02287	−0.02489	−0.02661	−0.02810
ω_{d3}	1.31778	1.56911	1.78442	1.97601	2.15041	2.31158	2.46216	2.60400	2.73847	2.86663
δ_{d3}	−0.00001	−0.00005	−0.00012	−0.00019	−0.00026	−0.00033	−0.00039	−0.00044	−0.00048	−0.00052

ω_{ex}—Angular frequency from the exact solution using Eq. (8).
ω_{d2}—Angular frequency from the two-divisions technique using Eqs. (14) and (22).
δ_{d2}—Percentage error defined by $\delta_{d2} = 100 \times (\omega_{d2} - \omega_{ex})/\omega_{ex}$.
ω_{d3}—Angular frequency from the three-divisions technique using Eqs. (15) and (24).
δ_{d3}—Percentage error defined by $\delta_{d3} = 100 \times (\omega_{d3} - \omega_{ex})/\omega_{ex}$.

If three-divisions technique shown by Eq. (15) is used, we have the following angular frequency:

$$\omega_{d3} = \frac{6Aq_1q_2q_3}{A^2(q_2q_3 + 2q_1q_3 + 2q_1q_2) + q_1q_2q_3q_4}, \tag{31}$$

where

$$q_1 = p_1, \quad q_2 = \sqrt{A^2 - (4/3)\ln[(1 + A^2)/(1 + A^2/4)]}$$
$$q_3 = \sqrt{A^2 - 4\ln[(1 + A^2)/(1 + 3A^2/4)]}, \quad q_4 = p_3. \tag{32}$$

The computed results for ω_{ex}, ω_{d2} and ω_{d3} are listed in Table 2. From listed results we see that the maximum error for ω_{d2} (ω_{d3}) is -0.2622% (-0.0612%), respectively

For this problem, three previously obtained solutions are cited below [9,10,14]

$$\omega^2 = \frac{3A^2}{4 + 3A^2} \quad \text{(latter, the relevant result is denoted by } \omega_{\text{th1}}), \tag{33}$$

$$\omega^2 = \frac{\lambda A^2}{1 + \lambda A^2} \text{ with } \lambda = (0.8472)^2 = 0.7177 \quad \text{(latter, the relevant result is denoted by } \omega_{\text{th2}}), \tag{34}$$

$$\omega^2 = 1 + \frac{2}{A^2}\left(\frac{1}{\sqrt{1 + A^2}} - 1\right) \quad \text{(latter, the relevant result is denoted by } \omega_{\text{th3}}). \tag{35}$$

The computed results for ω_{th1}, ω_{th2} and ω_{th3} are also listed in Table 2. The computed results for ω_{th1}, ω_{th2} and ω_{th3} are sufficient accurate for example in the range, for example, $A > 10$. However, they are not accurate, for example at $A = 1$. In the case of $A = 1$, the errors for ω_{d2}, ω_{d3}, ω_{th1}, ω_{th2} and ω_{th3} are -0.1118%, -0.0056%, 2.8086%, 1.5118% and 1.0701%, respectively. In Table 2, many places for $|\delta_{\text{th1}}| > 0.5\%$,

Table 2
Comparison results for the angular frequencies ω_{ex}, ω_{d2}, ω_{d3}, ω_{th1}, ω_{th2} and ω_{th3} for the Duffing-harmonic oscillation $d^2u/dt^2 + u^3/(1 + u^2) = 0$ with the conditions $u(0) = A$ and $u'(0) = 0$, for $A = 0.01, 0.05, 0.1, 0.5 \ldots, 10, 50, 100$

A	0.01000	0.05000	0.10000	0.50000	1.00000	5.00000	10.0000	50.0000	100.000
ω_{ex}	0.00847	0.04232	0.08439	0.38737	0.63678	0.96698	0.99092	0.99961	0.99990
ω_{d2}	0.00847	0.04230	0.08435	0.38712	0.63607	0.96462	0.98944	0.99942	0.99984
δ_{d2}	−0.04767	−0.04784	−0.04832	−0.06406	−0.11175	−0.24349	−0.14925	−0.01883	−0.00633
ω_{d3}	0.00847	0.04232	0.08439	0.38736	0.63674	0.96627	0.99031	0.99951	0.99986
δ_{d3}	−0.00115	−0.00117	−0.00119	−0.00194	−0.00562	−0.07252	−0.06121	−0.01023	−0.00361
ω_{th1}	0.00866	0.04326	0.08628	0.39736	0.65465	0.97435	0.99340	0.99973	0.99993
δ_{th1}	2.22071*	2.22562*	2.24077*	2.57996*	2.80683*	0.76309*	0.25057	0.01257	0.00323
ω_{th2}	0.00847	0.04232	0.08442	0.39005	0.64641	0.97325	0.99311	0.99972	0.99993
δ_{th2}	−0.00117	0.00750	0.03430	0.69254*	1.51182*	0.64865*	0.22094	0.01137	0.00293
ω_{th3}	0.00866	0.04326	0.08624	0.39423	0.64359	0.96731	0.99095	0.99961	0.99990
δ_{th3}	2.22029*	2.21501*	2.19869*	1.77209*	1.07006*	0.03459	0.00381	0.00001	0.00000

ω_{ex}—Angular frequency from the exact solution using Eq. (8).
ω_{d2}—Angular frequency from the two-divisions technique using Eqs. (14) and (29).
δ_{d2}—Percentage error defined by $\delta_{d2} = 100 \times (\omega_{d2} - \omega_{ex})/\omega_{ex}$.
ω_{d3}—Angular frequency from the three-divisions technique using Eqs. (15) and (31).
δ_{d3}—Percentage error defined by $\delta_{d3} = 100 \times (\omega_{d3} - \omega_{ex})/\omega_{ex}$.
ω_{th1}—Angular frequency from Eq. (33).
δ_{th1}—Percentage error defined by $\delta_{d2} = 100 \times (\omega_{\text{th1}} - \omega_{ex})/\omega_{ex}$.
ω_{th2}—Angular frequency from Eq. (34).
δ_{th2}—Percentage error defined by $\delta_{\text{th2}} = 100 \times (\omega_{\text{th2}} - \omega_{ex})/\omega_{ex}$.
ω_{th3}—Angular frequency from Eq. (35).
δ_{th3}—Percentage error defined by $\delta_{\text{th3}} = 100 \times (\omega_{\text{th3}} - \omega_{ex})/\omega_{ex}$.
*The error $|\delta| > 0.5\%$.

$|\delta_{th2}|>0.5\%$, $|\delta_{th3}|>0.5\%$, (δ_{th1}, δ_{th1} and δ_{th1} are the error for ω_{th1}, ω_{th2} and ω_{th3}) have been found. However, the errors for ω_{d2} and ω_{d2} are rather small.

3. Solution for the eardrum-type oscillation

In the following, we consider the eardrum-type oscillation [4,12]:

$$\frac{d^2u}{dt^2}+u+\varepsilon u^2=0 \quad (\text{or with } f(u)=u+\varepsilon u^2 \text{ in Eq. (1)}). \tag{36}$$

In this case, the restoring force $f(u)$ is not an odd function with respect to u, and the analysis is not very the same as in the previous section.

From Eq. (5) we have

$$g(u)=2\int_0^u f(u)\,du=u^2+\frac{2\varepsilon}{3}u^3. \tag{37}$$

Eq. (3), or $v^2+g(u)=g(A)$, is still valid in the present case. In the present case, another pair of the displacement and velocity, or $u=B$ and $v=0$, is the solution of Eq. (3). Substituting of $u=B$ and $v=0$ into Eq. (3) yields

$$B^2+\frac{2\varepsilon}{3}B^3=A^2+\frac{2\varepsilon}{3}A^3. \tag{38}$$

From Eq. (38), we can obtain a solution for B as

$$B=\frac{-(3+2\varepsilon A)+\sqrt{3(3+2\varepsilon A)(1-2\varepsilon A)}}{4\varepsilon}. \tag{39}$$

It is proved that, B is a negative value ($B<0$), and $-B>A$ is valid in general. Since B is a real value, $1-2\varepsilon A$ must be positive. Thus, the following condition should be satisfied:

$$1-2\varepsilon A>0 \quad \text{or} \quad 2\varepsilon A<1. \tag{40}$$

From Eq. (3), or $v^2+g(u)=g(A)$, we can obtain the period of the motion [1,2,10]

$$T_p=2\pi I_o=\frac{2\pi}{\omega} \quad \left(\text{with } I_o=\frac{1}{\omega}=\frac{1}{\pi}\int_B^A \frac{du}{\sqrt{g(A)-g(u)}}\right). \tag{41}$$

In the example, we assume $A=0.45$ and $\varepsilon=0.1, 0.2, \dots, 1.0$, the exact results from Eq. (41) are expressed as

$$\omega_{ex}=F_4(\varepsilon). \tag{42}$$

This result is obtained from Eq. (41) (using a substitution $u=s+D$ mentioned below), and a quadrture rule described in Appendix A [13].

In the following, we define

$$C=\frac{A-B}{2}, \quad D=\frac{A+B}{2} \tag{43}$$

and let

$$u=s+D, \quad g_1(s)=g(u)\big|_{u=s+D}=(s+D)^2+\frac{2\varepsilon}{3}(s+D)^3. \tag{44}$$

Thus, the integral I_o can be reduced to

$$I_o=\frac{1}{\omega}=\frac{1}{\pi}\int_{-C}^{C}\frac{ds}{\sqrt{g(A)-g_1(s)}} \tag{45}$$

or

$$I_o=\frac{1}{\omega}=\frac{1}{\pi}\int_{-C}^{C}\frac{h_1(s)ds}{\sqrt{C^2-s^2}} \quad \left(\text{with } h_1(s)=\frac{\sqrt{C^2-s^2}}{\sqrt{g(A)-g_1(s)}}\right) \tag{46}$$

Table 3
Comparison results for the angular frequencies ω_{ex}, ω_{d2}, and ω_{d3}, for the eardrum oscillation $d^2u/dt^2 + u(1+\varepsilon u) = 0$ with the conditions $u(0) = A$ and $u'(0) = 0$, for $\varepsilon = 0.1, 0.2, \ldots, 1.0$ and $A = 0.45$

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
ω_{ex}	0.99913	0.99639	0.99152	0.98418	0.97387	0.95981	0.94078	0.91463	0.87689	0.81531
ω_{d2}	0.99913	0.99639	0.99152	0.98418	0.97387	0.95981	0.94078	0.91463	0.87689	0.81530
δ_{d2}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00002	−0.00015	−0.00143
ω_{d3}	0.99913	0.99639	0.99152	0.98418	0.97387	0.95981	0.94078	0.91463	0.87689	0.81531
δ_{d3}	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	−0.00001

ω_{ex}, ω_{d2}, δ_{d2}, ω_{d3}, δ_{d3}, see notations in Table 1.

In Eq. (46), one can let $s = C\sin\theta(-\pi/2 \leqslant \theta \leqslant \pi/2)$. Two approximations are suggested below. If four divisions $(-\pi/2, -\pi/4)$, $(-\pi/4, 0)$, $(0, \pi/4)$ and $(\pi/4, \pi/2)$ are chosen for the interval $-\pi/2 \leqslant \theta \leqslant \pi/2$, and the trapezoid quadrature rule is used [13], we have the following angular frequency

$$\omega_{d4} = \frac{8}{p_1 + 2(p_2 + p_3 + p_4) + p_5}, \tag{47}$$

where

$$p_1 = \frac{C}{\sqrt{-C(B+\varepsilon B^2)}}, \quad p_2 = \frac{C}{\sqrt{2[g(A) - g_1(-C/\sqrt{2})]}}, \quad p_3 = \frac{C}{\sqrt{[g(A) - g_1(0)]}},$$
$$p_4 = \frac{C}{\sqrt{2[g(A) - g_1(C/\sqrt{2})]}}, \quad p_5 = \frac{C}{\sqrt{C(A+\varepsilon A^2)}}. \tag{48}$$

In Eq. (47), the subscript $d4$ means the result is obtained from four-divisions technique.

If six divisions $(-\pi/2, -\pi/3)$, $(-\pi/3, -\pi/6)$, $(-\pi/6, 0)$, $(0, \pi/6)(\pi/6, \pi/3)$ and $(\pi/3, \pi/2)$ are chosen for the interval $-\pi/2 \leqslant \theta \leqslant \pi/2$, and the trapezoid quadrature rule is used, we have the following angular frequency:

$$\omega_{d6} = \frac{12}{q_1 + 2(q_2 + q_3 + q_4 + q_5 + q_6) + q_7}, \tag{49}$$

where

$$q_1 = p_1, \quad q_2 = \frac{C}{\sqrt{4[g(A) - g_1(-\sqrt{3}C/2)]}}, \quad q_3 = \frac{C}{\sqrt{4[g(A) - g_1(-C/2)]/3}}, \quad q_4 = p_3,$$
$$q_5 = \frac{C}{\sqrt{4[g(A) - g_1(C/2)]/3}}, \quad q_6 = \frac{C}{\sqrt{4[g(A) - g_1(\sqrt{3}C/2)]}}, \quad q_7 = p_5. \tag{50}$$

In Eq. (49), the subscript $d6$ means the result is obtained from six-divisions technique. The computed results for ω_{ex}, ω_{d2} and ω_{d3} are listed in Table 3. From listed results we see that the maximum error for ω_{d2} (ω_{d3}) is −0.0014% (0.0000%), respectively.

4. Remarks

In this paper, total four examples are presented. The maximum percentage errors in four examples are 0.0002%, −0.0005%, −0.0725%, and 0.0000% (from Eq. (18) and Tables 1–3), respectively. The background for obtaining so high accuracy can be explained by the following reason.

It is an essential step to convert the integral shown by

$$I_o = \frac{2}{\pi}\int_0^A \frac{du}{\sqrt{g(A) - g(u)}} \tag{10}$$

into the form of Eq. (11)

$$I_o = \frac{2}{\pi}\int_0^A \frac{h(u)\,\mathrm{d}u}{\sqrt{A^2-u^2}}, \quad \text{where} \quad h(u) = \frac{\sqrt{A^2-u^2}}{\sqrt{g(A)-g(u)}}. \tag{11}$$

We know that in the harmonic motion case, there are $g(u) = u^2$, $h(u) = 1$, $I_o = 1$. In the nonlinear vibration case, the function $h(u)$ can be considered as a modification of the nonlinear vibration to the harmonic vibration. Therefore, the function $h(u)$ may be varied smoothly within the interval $(0,A)$. This good behavior of the function $h(u)$ must provide rather accurate result for the angular frequency.

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Appendix A. A quadrature rule for the integral shown by Eq. (8) [13]

The integral in Eq. (8) is written as

$$J = \int_{-A}^{A} \frac{\mathrm{d}u}{\sqrt{g(A)-g(u)}}. \tag{A.1}$$

The integral can be rewritten in the following form:

$$J = \int_{-A}^{A} \frac{h(u)\,\mathrm{d}u}{\sqrt{A^2-u^2}} \quad \text{with} \quad h(u) = \frac{\sqrt{A^2-u^2}}{\sqrt{g(A)-g(u)}}. \tag{A.2}$$

Meantime, the following quadrature rule was available [13]:

$$J = \frac{\pi}{M}\sum_{j=1}^{M} h(u_j), \quad \text{with } u_j = A\cos((j-0.5)\pi/M)), \quad \text{M-integer.} \tag{A.3}$$

In the present study, $M = 180$ was chosen. In this case, the computed result must be very accurate.

Appendix B. The trapezoid quadrature rule [13]

For the usage of the trapezoid quadrature rule, from Eq. (10c) the following integral is introduced:

$$K = \int_0^{\pi/2} h_1(\theta)\,\mathrm{d}\theta. \tag{B.1}$$

The integration interval $(0,\pi/2)$ may be divided into n divisions, and the width of the division will be $b = \pi/2n$. Therefore, n divisions (θ_i, θ_{i+1}) $(i = 1,2,\ldots,n)$ will be obtained, where $\theta_i = (i-1)b$, $(i = 1,2,\ldots,n+1)$ and $\theta_1 = 0$, $\theta_{n+1} = \pi/2$.

The contribution from the interval (θ_i, θ_{i+1}) to the integral will be

$$K_i = b\left(\frac{h_1(\theta_i)+h_1(\theta_{i+1})}{2}\right) = \frac{\pi}{4n}(h_1(\theta_i)+h_1(\theta_{i+1})). \tag{B.2}$$

Therefore, the integral defined by Eq. (B.1) can be integrated as follows:

$$K = \sum_{i=1}^{n} K_i = \frac{\pi}{4n}[h_1(\theta_1) + 2(h_1(\theta_2)+\cdots+h_1(\theta_n)) + h_1(\theta_{n+1})] \tag{B.3}$$

or

$$K = \sum_{i=1}^{n} K_i = \frac{\pi}{4n}\left[h_1(0) + 2(h_1(\pi/2n)+\cdots+h_1((n-1)\pi/2n)) + h_1(\pi/2)\right]. \tag{B.4}$$

In the case of $n = 2$, we have

$$K = \frac{\pi}{8}\left(h_1(0) + 2h_1(\pi/4) + h_1(\pi/2)\right). \tag{B.5}$$

In the case of $n = 3$, we have

$$K = \frac{\pi}{12}\left(h_1(0) + 2h_1(\pi/6) + 2h_1(\pi/3) + h_1(\pi/2)\right). \tag{B.6}$$

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